23.15
5.25
17.40
17.90
0.30

0.15
0.75
0.65
3.00
4.55
65
5.20

$P(A+B) = \frac{15}{34}$

$P(AB) = \frac{1}{17}$

$\frac{15}{34} + \frac{1}{17} = \frac{15+2}{34} = \frac{17}{34} = \frac{1}{2}$

B.V.S. Murthy

(*) Ref. is made in this paper to 3(i) and also to

(a) Proc. XI Intern. Congress of Appl. Mechanics, Springer-Verlag (available huge Vol. in [illegible]. No. 183 - First floor - Page Nos not given - found after a colossal search the page no. to be p. 131 (in the section on mechanics of solids

(b) [On p. 973 there an article by H. W. Liepmann, (Caltech) R. Narasimha (I.I.Sc) & M. Chahine (Caltech) on a shock structure problem. - Also other references to Narasimha

(c) Found on p. 1101 the name of B. R. Seth as Chairman of some Section

References to Ph.D. Thesis :

(1) Milne-Thomson – Theoretical Hydrodynamics, 2nd Edn (1950), p. 576

(2) S. C. Cowin – Polar fluids – Phys. of fluids 11, No. 9 (1968), p. ~~191~~ 1919.

(3) A. C. Eringen – (i) Simple microfluids – Int. J. Sci & Engg. 2 (1964), p. 205

(ii) Theory of micropolar fluids – J. Math & Mech. 16 (1966), p.

No (3)(ii) Available in No. 57 downstairs & (3)(i) first floor No. 191

No (2). available in ~~191~~ 62 on the ~~top~~ down floor.

(22/4/70)

Journal of Math. & Mech, 1966, p. 1 – Theory of micropolar fluids by A. C. Eringen ⊛

Abstract – Micropolar fluids respond to micro-rotational motions and spin inertia, and therefore, can support couple stress and distributed body couples. Thermodynamical restrictions are studied in detail and field equations are obtained for the density, velocity vector and micro-rotation vector. The system is solved for a channel flow exhibiting certain interesting phenomena. (27/4/70)

1. Introduction – A class of fluids which exhibit certain microscopic effects arising from the local structure and micromotions of the fluid elements – Theory of micro-fluids is, however, too complicated even in the case of constitutively linear theory, and the underlying mathematical problem is not easily amenable to the solution of non-trivial problems in this field.

A subclass of these fluids is the micropolar fluids which exhibit the microrotational effects and micro-rotational inertia. This class possesses certain simplicity & elegance in their mathematical formulation which should appeal to mathematicians. Physically they may represent adequately the fluids consisting of bar-like elements. Certain anisotropic fluids, eg. liquid crystals which are made up of ~~dumb~~ dumbbell molecules are of this type. In fact animal ~~blood~~ blood happens to fall into this category.

Recent expts. with fluids containing extremely small amounts of polymeric additives indicate that the skin friction near a rigid body in such fluids is considerably lower (upto 30-35%) than the same fluids without additives. The classical Navier-Stokes theory is incapable of predicting these findings since it contains no

mechanism to explain this new physical phenomenon. Suggestion of the microfluid theory as containing the right kind of mechanism

§ 2-3 – A resumé of the theory of microfluids formulated in (3)(i)

§ 4 – Theory of micropolar fluids

§ 5 – Thermodynamics of such fluids & restrictions on viscosity coefficients

§ 6 – Field eqns, boundary conditions & similarity parameters.

§ 7 – Soln of the problem of Channel flow for micropolar fluids.

Simple microfluids by Eringen in Int. J. Sc. & Engg. 2 (1964), p. 205

Introduction. Reference to a companion paper [ibid, p. 189] giving a non-linear theory for an elastic solid in which the first stress moments, micro-stress averages and inertial spin play important roles. Elastic solids exhibiting such local effects were named "simple micro-elastic material." In the present paper we investigate a new class of fluids which exhibit similar micro-effects

A simple micro-fluid, roughly speaking, is a fluent medium whose properties and behaviour are affected by the local motions of the material particles contained in each of its volume element [Precise definition given in §4 is

"A fluent medium is called a simple micro-fluid if it possesses equations of the form

$$\left.\begin{aligned} t_{kl} &= f_{kl}(v_{p;q}, v_{pq}, v_{pq;r}) \\ s_{kl} &= g_{kl}(\quad\quad\quad

deformation rate tensor, and various micro-deformation rate tensors. Fluids having surface tensions, anisotropic fluids, vortex fluids and fluids in which other gyrational effects are important are conjectured to fall into the domain of simple micro-fluids.

The simple micro-fluids are viscous fluids, and in the simplest case of constitutively linear theory contain 22 viscosity coefficients. Nonlinear Stokesian fluids turn out to be a special class of simple micro-fluids

In §2 - Discussion of motion and micromotion - Derivation and discussion of the gyration tensor, inertial spin and the conservation of micro-inertia, and objectivity of micro-deformation rate tensors.

§3. — Equations of balance, jump conditions, discussion of entropy productions and other relevant thermodynamic concepts.

§4 - Theory of the constitutive equations.

§4 - Partial diff. eqns of constitutively linear simple micro-fluids.

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§2. Theorem 1 - Micro-inertia moments i^{km} satisfy the following partial diff. eqs

$$\frac{\partial i^{km}}{\partial t} + i^{km}_{r;}\, v^r - i^{rm}\, v^k_{\ r} - i^{kr}\, v_r^{\ m} = 0 \qquad (2.16)$$

(v = gyration tensor)

This is the theorem of "conservation of micro-inertia moments"

Theorem 2 - The material derivative of micro-displacement differentials $d\xi^k$ is given by

$$\frac{D}{Dt}(d\xi^k) = v^k_{\ \ell}\, d\xi^\ell + v^k_{\ \ell;m}\, \xi^\ell\, dx^m \qquad (2.18)$$

Theorem 3 - The material derivative of the square of the arc length in the deformed body is given by

$$\frac{D}{Dt}(ds'^2) = [\ \ldots\ ]dx^k dx^\ell + 2(\quad)\, dx^k d\xi^\ell + (v_{k\ell} + v_{\ell k})\, d\xi^k d\xi^\ell \qquad (2.24)$$

deformation rate tensor $\underline{\underline{d}}$ is $d_{k\ell} \equiv \frac{1}{2}(v_{k;\ell} + v_{\ell;k}) = v_{(k;\ell)}$

micro-deformation rate tensor $\underline{\underline{b}}$ is $b_{k\ell} \equiv v_{k\ell} + v_{\ell k}$

" $\underline{\underline{a}}$ is $a_{k\ell m} \equiv v_{k\ell;m}$

From Th. 3, follows

Theorem 4 - A necessary and sufficient condition for micro-rigid motion is that $\underline{\underline{b}} = \underline{\underline{d}} = 0$.

This theorem replaces the well-known Killing's theorem of differential geometry.

Defn of an objective tensor ∘ 74.5 – The micro-deformation rate tensors $\underline{\underline{b}}$ and $\underline{\underline{a}}$ are objective tensors.

§ 3. Equations of balance and entropy

Eqns re. principle of conservation of mass, principle of Conservation of micro-inertia moments leading to (2.16), and axioms of local balance of momentum and conservation of energy are all given, with explanation of notation used

Axiom – A simple micro-fluid is assumed to possess an internal energy function ϵ which depends solely on entropy η, special volume $1/\rho$, and the micro-inertia moments i^{Km} ie

$$\epsilon = \epsilon\left(\eta, \frac{1}{\rho}, i^{Km}\right) \qquad (3.9)$$

Defns of thermodynamic temperature, pressure and micro-pressure tensor (π)

$$\tau_i^{\ k} = 2\rho\, i^{Km}\, \pi_{iKm} \qquad (3.14)$$

is called the thermodynamic micro-stress tensor

Enumeration of the five dissipative forces contributing to the true rate of change of entropy – Gives the Clausius-Duhem inequality –

Since our intention here is not to enumerate closely the foundations of the thermodynamics or to dwell into the thermal problems, we leave the subject matter of entropy at this point.

§ 4. Constitutive equations

Defn – A fluent medium (already given above) of a micropolar fluid

Theorem 6 – A second order objective tensor can only be an even tensor function of odd order tensors. A third order objective tensor can only be an odd function of any objective third order tensor.

Theorem 7 – The constitutive eqns of simple micro-fluids are equivalent to

$$t_i^{\ k} = f_i^{\ k}\left(\underline{\underline{d}}, \underline{\underline{b}} - \underline{\underline{d}}, \underline{\underline{a}}\right), \quad s_i^{\ k} = g_i^{\ k}\left(\underline{\underline{d}}, \underline{\underline{b}} - \underline{\underline{d}}, a\right), \quad \lambda_{im}^{\ k} = h_{im}^{\ k}\left(\underline{\underline{d}}, \underline{\underline{b}} - \underline{\underline{d}}, \underline{\underline{a}}\right) \qquad (4.10)$$

where f, g, h are subject to conditions leading to 74.6.

Theorem 8 – All Stokesian fluids are included in the class of simple micro-fluids represented by the constitutive eqns (4.13), (4.15) and (4.16) subject to (4.20)

Theorem 9 – For special types of body and surface moments and for $\underline{\underline{d}} = \underline{\underline{b}}$, all motions of simple micro-fluids coincide with those of

Stokesian fluids

Linear theory – Introduction of such a theory leads to the fact that the total number of viscosity coefficients is 22.

§ 5. Partial diff. eqns of motion

These equations are obtained by adjoining the eqns of conservation of mass (3.1), and conservation of micro-inertia moments (3.2) to those obtained by substituting the constitutive eqns into the eqns of balance. Result for the linear theory gives

$1+6+3+9=19$ eqns to determine the 19 unknowns ρ, $i^{km}=i^{mk}$, v_i^k and v^k since the body force f_i and the body moment l_i^k are supposed to be given and σ_i^k is expressible in terms of i^{km} & v_i^k – Difficulties of boundary & initial value problems – From the partial differential eqns we obtain the Navier Stokes eqns as a particular case by taking suitable values for parameters. This is but one more verification of the fact that the Navier-Stokes eqns are special cases of those of simple micro-fluids.

(29/4/70)

J. Math & Mech. 1966 – p. 1 – Theory of micropolar fluids by Eringen. (Contd).

§ 2 – Theory of microfluids:

Conservation of mass:
$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \quad \text{in } \mathcal{V} \qquad (2.1)$$

Balance of momentum:
$$t_{kl,k} + \rho(f_l - \dot{v}_l) = 0 \quad \text{in } \mathcal{V} \qquad (2.2)$$

Balance of first stress moments:
$$t_{ml} - s_{ml} + \lambda_{klm,k} + \rho(l_{lm} - \dot{\sigma}_{lm}) = 0 \quad \text{in } \mathcal{V} \qquad (2.3)$$

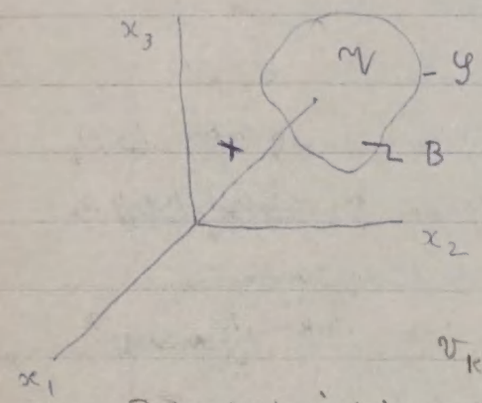
Conservation of energy:
$$\rho\dot{\epsilon} = t_{kl} v_{l,k} + (s_{kl} - t_{kl}) v_{kl} + \lambda_{klm} v_{ml,k} + q_{k,k} + \rho h \quad \text{in } \mathcal{V} \qquad (2.4)$$

Principle of entropy:
$$\rho\Gamma \equiv \rho\dot{\eta} - \left(\frac{q_k}{\theta}\right)_{,k} - \frac{\rho h}{\theta} \geq 0 \quad \text{in } \mathcal{V} \qquad (2.5)$$

Inequality (2.5) is axiomatised to be valid for all independent processes. In these equations

ρ = mass density
v_k = velocity vector
t_{kl} = stress tensor
f_l = body force per unit mass
s_{kl} = microstress tensor
θ = temperature.
λ_{klm} = the first stress moments
l_{lm} = the first body moments per unit mass.
$\dot{\sigma}_{lm}$ = inertial spin
ϵ = internal energy density per unit mass
ν_{kl} = gyration tensor
q_k = heat vector directed outward of the body
h = heat source per unit mass
η = entropy per unit mass.

Throughout this paper we affix a rectangular coordinate system x_1, x_2, x_3 and the Eulerian representation (Fig. 1). All vectors and tensors are referred to a set of spatial rect. coord's & hence no need of covariant, contravariant or mixed components. An index followed by a comma represents partial differentiation w.r.t space variable x_K and a superposed dot indicates material differentiation, eg



$$v_{K,L} \equiv \frac{\partial v_K}{\partial x_L} \; ; \; \dot{v}_K = \frac{\partial v_K}{\partial t} + v_{K,L} v_L \qquad (2.6)$$

Repeated indices denote summation over the range (1, 2, 3).

For the spin inertia, we have the kinematical relation (Ref. 1, eqn (5.5)

$$\dot{\sigma}_{KL} \equiv i_{ML} (\dot{v}_{MK} + v_{NK} v_{MN}) \qquad (2.7)$$

where $i_{ML} = i_{LM}$ is called the micro-inertia moments and according to the law of conservation of micro-inertia, they satisfy the partial diff. eqn [Ref. 1, (2.1

$$\frac{\partial i_{KM}}{\partial t} + i_{KM,r} v_r - i_{rM} v_{rK} - i_{Kr} v_{rM} = 0 \text{ in } V \qquad (2.8)$$

Expressions (2.1) to (2.5) and (2.8) are valid at all parts of the body B having volume V, and surface S, except at a finite number of discontinuity surfaces, lines and points. At the surface S of the body, we have the boundary conditions

$$t_{KL} n_K = t_L \text{ on } S \qquad (2.9)$$

$$\lambda_{KLM} n_K = \lambda_{LM} \text{ on } S \qquad (2.10)$$

where $\underline{n}$ is the exterior normal to S, and t_L and λ_{LM} are respectively the surface tractions and surface moments acting on S.

We note that while equations (2.1), (2.2) are well-known from the classical continuum mechanics, eqns (2.3), (2.4) and (2.8) are new. The first two of these equations reduce to classical results [Ref. to Eringen's book on "Non-linear theory of continuous media, McGraw Hill, 1962]

$$t_{KL} = t_{LK} \qquad (2.11)$$

$$\rho \dot{\epsilon} = t_{KL} v_{L,K} + q_{K,K} + \rho h$$

when $\lambda_{KLM} = l_{LM} = \nu_{LM} = 0$. Eqn (2.3) is, however, much more general than (2.11) and is the result of the new principle of balance of first stress moments as against the limited axiom of balance of moment of momentum of the classical theory. Eqns (2.8) have, however, no counterpart in the classical

If we exclude the heat conduction phenomena, in the present theory, the determination of motion requires the determination of the 19 unknown

$$\rho(\underline{x},t),\ \epsilon_{klm}(\underline{x},t),\ v_k(\underline{x},t),\ \nu_{kl}(\underline{x},t) \qquad [\epsilon_{klm} = \epsilon_{mlk} \rightarrow 6 \text{ unknowns}, \ \nu_{kl} \rightarrow 9,\ v_k \rightarrow 3,\ \rho \rightarrow 1]$$

as against the four unknowns v_k and ρ of the classical theory

§ 3. Constitutive eqns for micro-fluids. In ref. 1, we also gave a set of constitutive eqns for microfluids. For a non-heat conducting medium, these are enforced as relations between the $\{t_{kl}, s_{kl}, \tau_{klm}\}$ and the objective quantities

$$d_{kl} \equiv \frac{1}{2}(v_{k,l} + v_{l,k}) \qquad (3.1)$$

$$b_{kl} \equiv v_{k,l} + \nu_{kl} \qquad (3.2)$$

$$a_{klm} \equiv \nu_{kl,m} \qquad (3.3)$$

[(3.1) & (3.2) appear to be different from what they are in Ref. 1]

and ρ and ϵ_{klm}. Of these $\underline{\underline{d}}$ is the rate of deformation tensor, $\underline{\underline{b}}$ and $\underline{\underline{a}}$ are two new tensors respectively called micro-deformation rate tensor of second order, and its gyration gradient. Both of these latter quantities transform like absolute tensors under any rigid motion of the frame of reference, ie they are objective. Hence they are suitable for use as the independent constitutive variables.

For the present work we produce here only the results of the linear constitutive theory of micro-isotropic fluids (ie $\epsilon_{klm} = \epsilon\,\delta_{lm}$)

$$\underline{\underline{t}} = +\left[-\Pi + \lambda\,\mathrm{tr}\,\underline{\underline{d}} + \lambda_0\,\mathrm{tr}(\underline{\underline{b}} - \underline{\underline{d}})\right]\underline{\underline{I}} + 2\mu\,\underline{\underline{d}} + 2\mu_0(\underline{\underline{b}} - \underline{\underline{d}}) + 2\mu_1(\underline{\underline{b}}^T - \underline{\underline{d}}) \qquad (3.4)$$

$$\underline{\underline{s}} = \left[-\Pi + \lambda\,\overline{\mathrm{tr}}\,\underline{\underline{d}} + \eta_0\,\mathrm{tr}(\underline{\underline{b}} - \underline{\underline{d}})\right]\underline{\underline{I}} + 2\mu\,\underline{\underline{d}} + \xi_1(\underline{\underline{b}} + \underline{\underline{b}}^T - \underline{\underline{d}}) \qquad (3.5)$$

$$\tau_{klm} = (\gamma_1 a_{mrr} + \gamma_2 a_{rmr} + \gamma_3 a_{rrm})\delta_{kl} + (\gamma_4 a_{lrr} + \gamma_5 a_{rlr} + \gamma_6 a_{rrl})\delta_{km} + (\gamma_7 a_{krr} + \gamma_8 a_{rkr} + \gamma_9 a_{rrk})\delta_{lm} + \gamma_{10} a_{klm} + \gamma_{11} a_{kmc} + \gamma_{12} a_{lkm} + \gamma_{13} a_{mkl} + \gamma_{14} a_{lmk} + \gamma_{15} a_{mlk}$$

where $\underline{\underline{I}}$ is the unit tensor and $\lambda, \lambda_0, \mu, \mu_1, \eta_0, \xi_1$, and γ_1 to γ_{15} are the viscosity coefficients. Also tr denotes the trace and a superscript T indicates a transpose, eg

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{\mathrm{tr}}\,b_{kl} = b_{kk}, \quad b^T_{kl} = b_{lk}$$

The equation of state for these fluids can be given in the form

$$\epsilon = \hat{\epsilon}(\gamma, \rho^{-1}) \qquad (3.7)$$

where the thermodynamic pressure Π and the temperature θ are defined by

$$\Pi \equiv -\left.\frac{\partial \epsilon}{\partial \rho^{-1}}\right]_{\eta,\gamma} \quad ; \quad \theta = \left.\frac{\partial \epsilon}{\partial \eta}\right]_{\rho,\gamma} \qquad (3.8)$$

For non-heat conducting media, the nineteen unknowns (2.12) must satisfy the 13 partial diff. eqns obtained by substituting (2.7) and (3.4) to (3.6) into (2.1) to (2.3) and the six eqns (2.8) so that the number of independent equations are equal to that of unknowns. Equations so obtained are non-linear in the inertia terms and highly complicated otherwise. The purpose of the present paper is to give a new theory applicable to a large class of fluids falling within the framework of the micro-fluid theory formulated above, however possessing adequate mathematical simplicity to make the engineering problems tractable.

§4. Microfluids. A microfluid will be called microfluid if for all motions

$$\lambda_{k\ell m} = -\lambda_{kmL}, \quad \nu_{kL} = -\nu_{Lk} \qquad (4.1)$$

Microfluids exhibit only micro-rotational effects and can support surface and body couples. Fluid particles contained in a small volume element, in addition to the usual rigid motion, can rotate about the centroid of the volume element in an average sense described by the gyration tensor ν. No microstretch of particles is, however, allowed (ν_{kL} is skew-symmetric). Thus microfluids consist of a kind of dumbbell molecules (Refer to his paper at the Int. Congress on microfluids ?)

We now proceed to show that a class of microfluids satisfying (4.1) exists. The theory of such fluids is the subject of the remainder of the paper.

Proof of existence. If (4.1) is to be valid for all motions, we can deduce [using (3.6)]

$$\left.\begin{aligned} \gamma_1 - \gamma_2 + \gamma_6 - \gamma_9 &= 0 \\ \gamma_7 - \gamma_8 &= 0 \\ \gamma_{10} - \gamma_{12} + \gamma_{11} + \gamma_{13} &= 0 \end{aligned}\right\} \qquad (4.3)$$

Also $\quad m_{\ell r,r} + \epsilon_{k\ell r}\, t_{\ell r} + \rho(f_k - \dot{v}_k) = 0 \qquad (4.9)$

[$m_{kr} \equiv -\epsilon_{r\ell m}\lambda_{k\ell m}$ & $\epsilon_{r\ell m}$ is the alternating tensor]

The boundary conditions (2.10) can be replaced by $m_{rs}\, n_r = m_k$ on S — (4.13)

The eqn of energy may be replaced by (where $m_k = -\frac{1}{4}\epsilon_{klm}\lambda_{lm}$)

$$\rho\dot{e} = t_{kl}\left(v_{l,k} - \epsilon_{klr}\nu_r\right) + m_{kl}\nu_{l,k} + q_{k,k} + \rho h \qquad (4.16)$$

Constitutive eqns can be put in

$$t_{kl} = t_0\left(-\pi + \lambda_\nu v_{r,r}\right)\delta_{kl} + \mu_\nu\left(v_{k,l} + v_{l,k}\right) + \kappa_\nu\left(v_{l,k} - \epsilon_{klr}\nu_r\right) \qquad (4.14)$$

and $$m_{kl} = \alpha_\nu \nu_{r,r}\delta_{kl} + \beta_\nu \nu_{k,l} + \gamma_\nu \nu_{l,k} \qquad (4.17)$$

The assumptions of micro-isotropy and the skew-symmetry of ν_{kl} when used in (I·8) gives

$$\frac{D\ell}{Dt} = 0, \quad \text{or} \quad i = \text{const} = j/\rho \text{ on material lines}$$

Finally we get an expression for the inertial reaction

$$\dot{\sigma}_r = -\epsilon_{rkl}\dot{\sigma}_{kl} = -\epsilon_{rkl}\, i\left(\dot{\nu}_{ik} + \nu_{nk}\nu_{in}\right)$$

and using $(4.1)_2$, this reduces to

$$\dot{\sigma}_{ik} = j\dot{\nu}_k \qquad (4.21)$$

Summarising these results: Basic eqns of motion (2.1), (2.2), (4.9), energy (4.10) and the constitutive eqns (4.14) and (4.17) constitute a proof that the micropolar fluids may exist as a subclass of microfluids whenever (4.3) is satisfied

§5. Thermodynamics of micropolar fluids — Gives two theorems using the 2nd Law of Thermodynamics

§6. Field equations — The diff. eqns satisfied by ρ, v_i and ν_k are given by (2.1) and combinations of (4.14) and (4.17) with (2.2) and (4.9), i.e

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \qquad (6.1)$$

$$-\pi_{,k} + (\lambda_\nu + \mu_\nu)v_{l,kl} + (\mu_\nu + \kappa_\nu)v_{k,ll} + \kappa_\nu \epsilon_{klm}\nu_{m,l} + \rho(f_k - \dot{v}_k) = 0 \qquad (6.2)$$

$$(\alpha_\nu + \beta_\nu)\nu_{l,kl} + \gamma_\nu \nu_{k,ll} + \kappa_\nu \epsilon_{klm} v_{m,l} - 2\kappa_\nu \nu_k + \rho(\ell_k - j\dot{\nu}_k) = 0 \qquad (6.3)$$

where a superposed dot indicates the material differentiation i.e

$$\dot{v}_k \equiv \frac{\partial v_k}{\partial t} + v_{k,l}v_l, \quad \dot{\nu}_k \equiv \frac{\partial \nu_k}{\partial t} + \nu_{k,l}v_l \qquad (6.4)$$

[This is same as (2.6) - why repeat?]

The partial differential eqns (6.1) to (6.3) are the field equations of the micro-polar fluids. Under appropriate initial and boundary conditions they must be capable of predicting the behaviour of such fluids in full. The existence and uniqueness theorems must, of course, be proven in order for the underlying mathematical problem to be "well posed". Presently we only have a few isolated initial and boundary conditions.

Initial conditions at $t = 0$

$$\left.\begin{aligned} \rho(\underline{x}, 0) &= \rho_0(\underline{x}) \\ v_k(\underline{x}, 0) &= v_{0k}(\underline{x}) \\ \nu_k(\underline{x}, 0) &= \nu_{0k}(\underline{x}) \end{aligned}\right\} \qquad (6.5)$$

$\rho = \rho_0 + \underline{\underline{\nu}}_0$ and $\underline{\underline{\nu}}_0$ are to be prescribed throughout

Boundary conditions at a rigid boundary

$$\left.\begin{aligned} \underline{V}(\underline{X}_B, t) &= \underline{V}_B \\ \underline{\nu}(\underline{X}_B, t) &= \underline{\nu}_B \end{aligned}\right\} \qquad (6.6)$$

where $\underline{X}_B$ is a point on a rigid boundary having prescribed velocity $\underline{V}_B$ and prescribed microrotation vector $\underline{\nu}_B$. Conds (6.6) expres the assumption of adherence of the fluid to the rigid boundary.

Boundary conditions involving prescribed forces and moments

In place of (6.6) we may prescribe boundary forces and moments as expressed by (2.9) and (4.13) ie

$$\left.\begin{aligned} t_{kl}\, n_k &= t_l \\ m_{kl}\, n_k &= m_l \end{aligned}\right\} \qquad (6.7)$$

Other types of mixed conditions are possible. The final judgment on these questions requires theoretical work on the question of existence and uniqueness and experimental work on the flow conditions.

Eqns (6.1) to (6.3) are expressed in rectangular coordinates. Vect expressions of these equations useful for work in other systems of coords are

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{V}) = 0 \qquad (6.8)$$

$$(\lambda_\nu + 2\mu_\nu + K_\nu)\nabla\nabla\cdot V - (\mu_\nu + K_\nu)\nabla\times\nabla\times V + K_\nu \nabla\times\underline{\nu} - \nabla\pi + \rho f = \rho\left[\frac{\partial V}{\partial t} - V\times(\nabla\times V) + \frac{1}{2}\nabla(V^2)\right] \qquad (6.9)$$

$$(\alpha_\nu + \beta_\nu + \gamma_\nu)\nabla\nabla\cdot\underline{\nu} - \gamma_\nu \nabla\times\nabla\times\underline{\nu} + K_\nu \nabla\times V - 2K_\nu \underline{\nu} + \rho \underline{l} = \rho j \dot{\underline{\nu}} \qquad (6.10)$$

where $\dot{\underline{\nu}}$ does not possess as simple an expression as $\dot{V}$. There is, however, no particular difficulty in calculating it through its tensorial form [Cf. §-17 of Ref. Eringen's paper]

We note that for $K_\nu = \alpha_\nu = \beta_\nu = \gamma_\nu = 0$ and vanishing $\underline{l}$ through (6.3) we get $\nu_K = 0$ and (6.2) reduces to the celebrated Navier-Stokes eqns. Note also that $K_\nu = 0$ the velocity $\underline{V}$ and the micro-rotation are uncoupled and the global motion is unaffected by the micro-rotations.

The classical Stokes condition $3\lambda_\nu + 2\mu_\nu = 0$ for the micro-polar fluids have the corresponding form

$$3\lambda_\nu + 2\mu_\nu + K_\nu = 0$$

to which we place no great faith.

ρ_0

For incompressible fluid $\rho = \text{const}$, $\nabla \cdot \underline{v} = 0$, and Π is replaced by an indeterminate pressure p to be determined from the boundary conditions

The similarity parameters of the micro-polar fluids are obtained by non-dimensionalising (6.1) to (6.3). Thus let L and T be respectively some characteristic length and time, and

$$\left.\begin{aligned} \bar{x} &= x/L, \quad \bar{t} = t/T, \quad \bar{v} = v/v_0, \quad \bar{\nu} = \nu/\nu_0 \\ \bar{\Pi} &= \Pi/\Pi_0, \quad \bar{\rho} = \rho/\rho_0, \quad \bar{f} = f/f_0, \quad \bar{j} = j/j_0 \end{aligned}\right\} \qquad (6.12)$$

where Π_0, ρ_0, v_0, ν_0, f_0 and j_0 are some reference values of Π, ρ, $|v|$, $|\nu|$, $|f|$ and j respectively. Substituting (6.12) into (6.1) to (6.3) and using (6.4) we get the non-dimensional equations

$$n_5 \frac{\partial \bar{\rho}}{\partial t} + (\bar{\rho}\,\bar{v}_k)_{,k} = 0 \qquad (6.13)$$

$$n_1 \bar{v}_{l,kl} + n_2 \bar{v}_{k,ll} + n_3 \epsilon_{klm} \bar{\nu}_{m,l} - n_4 \bar{\Pi}_{,k} + \bar{\rho}\left(n_6 \bar{f}_k - n_5 \frac{\partial \bar{v}_k}{\partial t} - \bar{v}_{k,l}\bar{v}_l\right) = 0 \qquad (6.14)$$

$$m_1 \bar{\nu}_{l,kl} + m_2 \bar{\nu}_{k,ll} + m_3 \epsilon_{klm} \bar{v}_{m,l} - 2 m_4 \bar{\nu}_k + \bar{\rho} \bar{j}\left(n_6 \bar{l}_k - n_5 \frac{\partial \bar{\nu}}{\partial t} - \bar{v}_{k,l}\bar{\nu}_l\right) \qquad (6.15)$$

where

$$\left.\begin{aligned} n_1 &\equiv (\lambda_v + \mu_v)/\rho_0 v_0 L, \quad n_2 \equiv (\mu_v + K_v)/\rho_0 v_0 L, \quad n_3 \equiv K_v \nu_0/\rho_0 v_0^2 \\ n_4 &\equiv \Pi_0/\rho_0 v_0^2, \quad n_5 \equiv L/T v_0, \quad n_6 \equiv f_0 L/v_0^2 \\ m_1 &= (\alpha_v + \beta_v + \gamma_v)/\rho_0 j_0 \nu_0 L, \quad m_2 = \gamma_v/\rho_0 j_0 \nu_0^2, \quad m_3 = K_v/\rho_0 j_0 \nu_0 \\ m_4 &= K_v L/\rho_0 j_0 \nu_0, \quad m_5 \equiv n_5, \quad m_6 = l_0 L/j_0 \bar{j} \nu_0 v_0 \end{aligned}\right\} \qquad (6.16)$$

Of these n_1, n_2 are the reciprocal Reynolds numbers; n_4, n_5 & n_6 are well known from the Navier-Stokes theory. The present theory introduces five new numbers viz. n_3, m_1, m_2, m_3, m_4 and m_5*. For a given fluid m_1 is equal to m_2 so that the only new parameters are

$$\left.\begin{aligned} n_3 &= K_v \nu_0/\rho_0 v_0^2, \quad m_2 = \gamma_0/\rho_0 j_0 \nu_0, \quad m_4 = K_0 L/\rho_0 j \nu_0 \\ \bar{m}_3 &= m_4/m_3 = \nu_0 L/v_0; \quad m_6 = l_0 L/j_0 \bar{j} \nu_0 v_0 \end{aligned}\right\} \qquad (6.17)$$

The four of these new similarity parameters represent the relative importance of rotational viscosities to the inertia terms, and the fifth m_5 the relative micro-rotation velocity to the velocity

[* This statement is not correct in view of $m_5 = n_5$ i.e. the present theory introduces only five new numbers]

in fact only two ones since m_1 is $= m_2$

Finally an <u>undergraduate couple stress theory of fluids</u> may be obtained as a special case of the theory of <u>micropolar fluids</u> when the micro-rotation is <u>constrained</u> by

$$v_k = \omega_k \qquad \text{Eq. (6·18)}$$

The development here is parallel to the one given in [refers to some articles to be published in J. Math & Mech.] so that the details are omitted. In this case the stress is decomposed into two parts

$$t_{kl} = t_{(kl)} + t_{[kl]} \qquad (6.19)$$

where the symmetric part $t_{(kl)}$ is understood to be given by (4·16) subject to (6·17) and the skew-symmetric part $t_{[kl]}$ is calculated from (4·9) & (6·17) to

$$t_{[kl]} = \frac{\gamma_0}{4}\left(v_{k,lnn} - v_{l,knn}\right) + \rho\left(l_{[kl]} - \frac{j}{2}\frac{D}{Dt}\left(v_{[k,l]}\right)\right) \qquad (6.20)$$

Using (6·18) in (2·2) we obtain the <u>field equations of the theory</u>

$\lambda_0 \neq \mu_0$

$$\left(\lambda_0 + \mu_0 + \frac{K_0}{2} + \frac{\gamma_0}{4}\nabla^2\right)v_{k,lk} + \left(\mu_0 + \frac{K_0}{2} - \frac{\gamma_0}{4}\nabla^2\right)v_{l,kk} - \pi_{,l} + \rho f_l + \frac{\rho}{2}\left(l_{kl,k} - l_{lk,k}\right) = \rho\dot{v}_l + \frac{\rho j}{4}\left[\frac{D}{Dt}\left(v_{k,l} - v_{l,k}\right)\right]_{,k} \qquad (6.21)$$

where D/Dt is the material derivative operator.

§7. <u>Flow of micropolar fluids in a circular pipe</u> — obtains solns of field eqns (6·8) to (6·10) for a steady motion of micropolar fluids in a circular channel — uses cylindrical coordinates — uses Bessel fns of 1st & 2nd order — assumes that fluid sticks to the boundary at $r = a$ — uses the notion of the classical Poiseuille flow (which comes as a particular case) — Fig 3 is velocity profile for $\lambda = 1$ $\left[\lambda = ka = \left[\frac{2\mu_0 + K_0}{\mu_0 + K_0}\cdot\frac{K_0}{\gamma_0}\right]^{1/2} a\right]$

a is perhaps the radius of the circular section, the vel. profile is no longer parabolic — Fig 4 shows Adverse Microflow — Fig 5 shows Micro-rotation — mentions that velocity here is smaller than that of the classical Navier-Stokes fluids, since the micro-rotation is altogether absent in the N-S theory — Fig 6 shows Shear Stress difference — Fig 7 shows Couple Stress —

<u>Final remarks</u> :— On Figs 6 and 7 are shown the surface tensions and couple

Cf.

on the fluid surface adjacent to the wall at $\rho = 1$ [$\rho = r/a$, r, θ, z being cyl. coord] for $dp/dz \leq 0$, $\beta_v < 0$ and, of course $\gamma_0 \geq 0$. The shearing stress t_{rz} has the same expression as in the classical theory. However the existence of the disturbed couples $m_{\theta r}$ on the fluid surface (Fig 3) will produce an effect in a thin layer near the wall, which leads to reduction of the surface shear. Clearly then the present theory gives rise to a boundary layer phenomenon not present in the N-S theory. This new boundary layer is controlled by the parameter γ.

We believe that the theory of microfluids offers up a very worthwhile branch of fluid mechanics. It should find important applications dealing with a variety of fluids. It should, in particular, cast new directions in the theory of turbulence. Rich theoretical and experimental studies are awaiting the future workers.

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§1. The theory of microfluids deals with a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements. These fluids can support stress moments [body moments] and are influenced by the spin inertia. The theory of microfluids is however too complicated even in the case of constitutive equations and the underlying mathematical problem is not easily amenable to the non-trivial problems in this field.

A subclass of these fluids is the micropolar fluids which exhibit the micro-rotational effects and micro-rotational inertia. This class of fluids possesses certain simplicity and elegance in their mathematical formulation which should appeal to mathematicians. The micropolar fluids can support only symmetric stresses and body couples only. Physically they may represent adequately the fluids consisting of bar-like elements. Certain anisotropic fluids e.g. liquid crystals which are made up of dumbbell molecules are of this type. In fact, animal blood happens to fall into this category. Other polymeric fluids and fluids containing certain additives may be represented by the mathematical model underlying micropolar fluids.

Recent experiments [Refs to papers by Hoyt & Fabula*, and Vogel and Patterson**] with fluids containing extremely small amount of polymeric additives indicate that the skin friction near a rigid body in such bodies is lowered by τ_{f_∞}.

[* U. S. Naval Ord. Test Station Reports, 1964
** Pac. Naval Lab. of the Res. Bd of Canada, Rpt 64-2

(up to 30–35%) from the same fluids without additives. The classical N-S theory is incapable of predicting these findings since it contains no mechanism to explain this new physical phenomenon. At the Naval Hydrodynamic Conference at Bergen last year Sept 1964, the author [Proc. Sept. 10 1964] suggested that the microfluid theory may contain just the right mechanism required. While it is too early to make the final conclusion on this question, the problem of channel flow worked out in this paper is a positive indication of this conjecture.

Physics of fluids. Vol 11, No: 9, p. 1919.

Polar fluids by S. C. Cowin:

Abstract – It is noted that a number of theories proposed recently by various authors are actually identical with the theory of polar fluids suggested by Grad in 1952 [Comm. Pure & Appl Math 5, 455 (1952)]. The relation of close relationship of the recently presented theory of dipolar fluids to polar fluids theory is established. The thermodynamics of polar fluids with heat conduction is sketched. The solved boundary-value problem of polar fluid theory are summarized and the essential features of these solutions establish a pattern that gives insight into the behaviour of polar fluids as well as indicating when the theory might be of practical significance.

Introduction – In polar fluid kinematics the total or particle angular velocity differs from the regional angular velocity; this difference called the relative angular velocity, represents the velocity of the fluid particle relative to the rigid continuum in which it is embedded – Different viewpoints, Aero, Bulygin, Kunstadter, Cowin, Sadler and Eringen – Eringen obtained the constitutive equations for a polar fluid, which he calls a micropolar fluid by specializing the most general theory of micro fluids. In polar fluid theory there are two vector and independent kinematic vector fields introduced, the vector field representing the velocities of the field particles and an axial vector field representing the angular velocities of the fluid particles. In traditional kinematics one deals only the velocity field as independent and considers the angular velocity of a fluid particle to be coincident with the regional angular velocity, which is equal to one half of the curl of the velocity field] – There are other theories of fluid behaviour that can be related to polar fluid theory for eg. the theory of dipolar fluids due to Green and

Grad [Int. J. Engg. Sci. 2, 323, (1967)]. A theory proposed by Born Born (Z. f. Physik 1, 221, 1920] anticipated Grad, but did not fully develop the all the aspects — Contents of several sections — Sec VI develops the relationship between polar & ordinary fluid — Restrictive conditions

Section II — Kinematics & dynamics of polar fluids (all vectorial)

Section III — Polar fluids with heat conduction

Section IV — Special cases of flows (i) Creeping flows (ii) Another special case difficult to define simply

Section V — Boundary-value problems — Two of problems mentioned including Eringen's pipe flow [others like plane Couette flow, considered by Condiff and Dahler and Pennington, who have also considered plane Poiseuille flow, Couette flow between concentric cylinders and Poiseuille or pipe flow. Hero, Brulygin and Kuvshinski considered pipe flow, the translational motion of a rigid sphere in an infinite fluid and the viscosity of dilute suspensions] There are certain general characteristics of these flows that are significant. Firstly, the polar fluid will exhibit a resistance to motion greater than or equal to that of the corresponding viscous fluid in the traditional flow situations. Second, the resistance of the polar fluid to motion will increase as the length ratio L [defined as the ratio of a characteristic geometric length L_0 to the material characteristic length ℓ, i.e. $L = L_0/\ell$] decreases. — These points are illustrated by problems for orthogonal plane Couette flow (flow between parallel flat plates in relative motion) and steady pipe flow. — Figures for velocity distribution in both the cases are shown. — The characteristics of the solutions to these two boundary value problems are similar to all the solutions to other problems. Plane Poiseuille flow gives results which are very similar to those for pipe flow. Here Stokes' solution to the problem of an infinite sphere moving through an otherwise polar fluid shows an increase in the resistance in comparison with that given by Stokes' formula. The same authors' determination of the viscosity of a polar fluid with suspensions shows it to be greater than the value given by Einstein [Ann. Physik, 19, 289 (1906); 34, 591 (1911)] for the viscosity of a viscous fluid with suspensions].

Although the calculations presented above for pipe flow agree with those given by Eringen, exception is taken to Eringen's conclusion that swirl and shear is reduced in pipe flow of polar fluids over that in viscous fluids.

The shear stress that is less than that predicted by classical theory acts in a plane ⊥r to a shearing plane and not on a shearing plane. The resistance of the polar fluid to motion is in no sense decreased because this shear stress is less than that given by the classical viscous fluid theory. By extension, one feels is also taken to Eringen's conjecture that polar fluids might serve as a theoretical model for the exptl. results showing that polymeric additives reduce skin friction — Suggests that the polar fluid theory might have applications, for example, in lubrication theory — Summary of related exptl. literature given by Hennikes [Rev. Mod. Phys. 21, 322 (1949)] — Hennikes remarks that "the evidence for extremely high viscosity in the neighborhood of a solid surface is extensive" He mentions an experiment in which a tenfold increase in viscosity was found within 5000 Å of a solid boundary and one performed by several people in which a ten-fold increase in the viscosity of water between glass plates 2500 Å was observed.

Section VI — Dipolar fluids — Compares this with polar fluids & shows that in almost all cases the results of the two theories are identical — Looks almost a futile exercise in mathematics without any hydrodynamical meaning!

Aero, Bulygin and Kuvshinski J. Appl. Math Mech, 29, 333, 1965
(Available in No. 58 downstairs).

[Re. this work Cowin says "Aero, Bulygin and ~~Kuvshinski~~ Kuvshinski simply introduce the velocity and total angular velocity fields and postulate a dissipation function that leads to Grad's constitutive equations". [But these constitutive eqns are not referred to as due to Grad in Aero et al's paper - only a reference is made in the further to Grad's work in the introduction]].

Cowin Comm. Pure & Appl. Math 5, 455, 1952 — Statistical Mechanics, Thermodynamics and Fluid Dynamics of systems with an arbitrary number of integrals by Harold Grad (referred to in Cowin's paper & found in No. 56 downstairs) — Meaningless paper to understand what his constitutive equations are! [Contains references to Born & Green's book on a "general kinetic theory of liquids" Cambridge 1949 & to a book by Schrödinger "Statistical thermodynamics" Cambridge

also la Bocherb sfeltra, Chapman & Cowling, Courant (Coll & Int. Calc), Gibbs Collected works
article by P. R. Halmos & L. Truesdell & so on !]

Aero evals' article

Introduction - The so-called asymmetric mechanics of continuous media is a generalisation of the ordinary continuum mechanics of continuous media for the case in which the stress tensor becomes asymmetric ($\sigma_{ik} \neq \sigma_{ki}$)

The couple $\sigma_{ik} - \sigma_{iki} = \frac{1}{2}(\sigma_{ik} - \sigma_{ki})\,\varepsilon_{iki}$ defines the resultant of the moments of the forces which act on an element of the medium. Therefore, this couple [left out by Truesdell & de Corre] has existence only if the resultant may be balanced only by the moments of couples in a series of works [Ref. to works of Truesdell & de Corre] the existence of volume-distributed couples m_i, stipulated by external effects, is assumed. However, the case in which the couples act between particles of the medium is of greater interest [Ref. to papers by Jaffen & Stevenson, E. S. Rajagopal (Ann. Phys, Bd 6, No 3-4, p. 192, 1960 entitled "The existence of interfacial couples in infinitesimal elasticity", and by Kurashinski & Aero]. The effects of one contiguous part of the medium on another is often characterised not only by surface forces (stresses), but also by surface moments (micro-moments). This concept is reflected in a series of papers [K & A, A & K, Griole, Minlin and Tiersten]

Initially asymmetric mechanics of continuous media was developed as a theory of elasticity, but recent investigations have appeared [two papers by Dahler one in J. Chem. Phys 30, 1447, 1957 & another in Nature, 192, No 4797, 1961, and the paper by Grad in Comm. P & Ap. Maths] in which asymmetric hydrodynamics is worked out. The need for these investigations stems from the desire to define more precisely the limits of validity of the classical hydrodynamics of viscous media. Asymmetric hydrodynamics may possibly explain a series of experimental data from the predictions of the theory, or differs from ordinary hydrodynamics by a more precise definition of the state of stress which is characterised by an asymmetric stress tensor σ_{ik} ($\sigma_{ik} \neq \sigma_{ki}$) and a tensor of surface micro-moments μ_{ik}.

The state of deformation is described by the deformation rate tensor ε^*_{ik} and by the micro-torsion and the micro-bending rate tensor γ^*_{ik}. The rheological laws in the relation of σ_{ik} and μ_{ik} with ε^*_{ik} and γ^*_{ik} are established. Additional coefficients which may be called the rotational viscosity coefficients appear in these relations

The equations of motion of the fluid in the velocity-vorticity and their general solutions are constructed. Boundary conditions sufficient to solve the eqns of motion are formulated. Liquid discharge from a capillary, the motion of a sphere in the fluid and the viscosity of suspensions are considered.

§1. The stress tensor

§2. The dissipation function

§3. The rheological laws

§4. The basic equations. Neglecting the dependence of the coefficients of viscosity $(\lambda, \mu, \eta, \theta, \gamma)$ on the coordinates we obtain

$$\rho \frac{d\underline{V}}{dt} = \rho f - \operatorname{grad} p + (\lambda+2\mu)\operatorname{grad}\operatorname{div} \underline{V} - (\mu-\gamma)\operatorname{rot}\operatorname{rot}\underline{V} = 2\gamma \operatorname{rot}\Omega' \quad (4.1)$$

$$(\eta+\tau+\theta)\operatorname{grad}\operatorname{div}\Omega' - \theta\operatorname{rot}\operatorname{rot}\Omega' + 2\gamma\Omega' - \gamma\operatorname{rot}\underline{V} + \rho m = 0 \quad (4.2)$$

The continuity eqn has the same form as in ordinary hydrodynamics

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{V}) = 0 \quad (4.3)$$

Heat transfer is

$$\rho \frac{d\epsilon}{dt} = \operatorname{div}(\varkappa \operatorname{grad} T) + q\,(\sigma_{ik}\,\epsilon^{*}_{ik} + \mu_{ik}\,\gamma^{*}_{ik}) \quad (4.4)$$

• §5. General solution of eqns of motion for the case of "creeping" motions. — Restricting to the case of steady "creeping" motion (the Stokes approximation) applying the function div to (4.1) & (4.2) we obtain

$$\Delta q = 0, \quad k_1^2 \Delta\varphi - \varphi = 0 \quad (5.1)$$

where

$$q = (\lambda+2\mu)\operatorname{div}\underline{V} - p, \quad \varphi = \operatorname{div}\Omega' \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad (5.2)$$

$$k_1^2 = -\tfrac{1}{2}(\eta+\theta+\tau)\gamma^{-1} \quad (\gamma \neq 0)$$

Further substituting (4.2) with (4.1) we arrive at the equation

$$\operatorname{grad} q - 2\mu \operatorname{rot}\Omega' + \theta(\mu-\gamma)\gamma^{-1}\operatorname{rot}\operatorname{rot}\operatorname{rot}\Omega' = 0 \quad (5.3)$$

• — from which we easily find

$$\operatorname{rot}\Omega' = \Omega_1' + \tfrac{1}{2}\mu^{-1}\operatorname{grad} q \quad (5.4)$$

where Ω_1' satisfies the equation

$$k_2^2 \operatorname{rot}\operatorname{rot}\Omega_1' + \Omega_1' = 0, \quad k_2^2 = \tfrac{1}{2}\theta(\mu-\gamma)\mu^{-1}\gamma^{-1} \quad (5.5)$$

Taking (5.4) into consideration, we obtain from (4.1) instead of equation a separate equation for

$$-\operatorname{grad} p + (\lambda+2\mu)\operatorname{grad}\operatorname{div}\underline{V} - \mu\operatorname{rot}\operatorname{rot}\underline{V} = 2\gamma\mu(\mu-\gamma)^{-1}\Omega_1' \quad (5.6)$$

which may be regarded as inhomogeneous, the solution has the form

$$\underline{V} = \underline{V}^0 - k_1^2\mu^{-1}\Omega_1' \quad (5.7)$$

Here $\underline{V}^0$ is that vector which corresponds to the homogeneous equation

$$-\eta \operatorname{rot}\operatorname{rot}\underline{V}^0 + (\lambda + 2\mu)\operatorname{grad}\operatorname{div}\underline{V}^0 - \mu \operatorname{rot}\operatorname{rot}\underline{V}^0 = 0 \qquad (5.8)$$

i.e. the Navier-Stokes equation of moving hydromechanics, and $-\theta_\mu^{-1}\Omega'$ is the fundamental solution (5.6) which is easily found by (5.8) consider.

We shall now find Ω_1' : substituting (5.3) into (5.4) and using (5.8) we get the equation

$$\operatorname{rot}\Omega' = -K_2^2 \operatorname{rot}\operatorname{rot}\Omega_1' + \frac{1}{2}\operatorname{rot}\operatorname{rot}\underline{V}^0 \qquad (5.9)$$

from which we find

$$\Omega' = \frac{1}{2}\operatorname{rot}\underline{V}^0 - K_2^2 \operatorname{rot}\Omega_1' + \operatorname{grad}\varphi^* \qquad (5.10)$$

to within $\operatorname{grad}\varphi^*$ which for the present is undetermined. Equating expression $\operatorname{div}\Omega'$ found from (5.10) with (5.4) we get $\varphi^* = K_1^2\varphi$ to within an unessential harmonic function. Hence

$$\Omega_0' = \frac{1}{2}\operatorname{rot}\underline{V}^0 = K_2^2 \operatorname{rot}\Omega_1' + K_1^2 \operatorname{grad}\varphi \qquad (5.11)$$

Thus, the general axisymmetric equations system of equations for the theory of flow of a viscous fluid (4.1) & (4.2) are given by the relations (5.7) and (5.11) i.e. the solution of the well known system of the usual N-S eqn and of eqn of the Helmholtz type (5.1) and (5.5)

We note that the latter can be written in the form of a single equation introducing a variable single vector Ω_2' in place of φ and Ω_1' ($\operatorname{div}\Omega_1' = 0$)

$$\Omega_2' = -K_2^2 \operatorname{rot}\Omega_1' + K_1^2 \operatorname{grad}\varphi \qquad (5.12)$$

which satisfies the equation

$$K_1^2 \operatorname{grad}\operatorname{div}\Omega_2' - K_2^2 \operatorname{rot}\operatorname{rot}\Omega_2' - \Omega_2' = 0 \qquad (5.13)$$

Then in place of (5.7) and (5.11) we have

$$\underline{V} = \underline{V}^0 - \theta\mu^{-1}\operatorname{rot}\Omega_2' \qquad \Omega' = \Omega_2' + \frac{1}{2}\operatorname{rot}\underline{V}^0 \qquad (5.14)$$

§6. The boundary conditions.

§7. The discharge of fluid from a capillary

§8. The translational motion of a sphere.

We shall consider the problem of flow about a sphere of radius R located at the origin of the spherical system of coordinates (r, θ, φ), by a stream of viscous fluid which has a constant velocity u_i at ∞. The solving the Navier-Stokes, as is well known [Ref. to Landau & Lifschitz - Mechanics of continuous media]

$$v_i^0 = u_i + \varepsilon_{ipm}\varepsilon_{pks}\frac{\partial^2}{\partial x_m \partial x_s}\left[f(r)\,u_k\right] \qquad (6.1)$$

$$f(r) = ar + b/r$$

Here and in what follows summation convention is adopted. The vector (5.8) can be found in the form

$$\Omega'_{ic} = \varepsilon_{ipm}\varepsilon_{lks}\frac{\partial^2}{\partial x_r \partial x_s}\left[\gamma(r)\, u_l\right], \quad \gamma_r = \frac{\rho}{r}e^{-r/k_2} \qquad (8.2)$$

Substituting (8.1) and (8.2) into (5.7), we find the field of translational velocities

$$v_i = u_i\left[1+F_1(r)\right] + n_i(n_k u_k)F_2(r) \qquad (8.3)$$

$$\left.\begin{aligned} F_1(r) &= -\left\{\frac{a}{r}+\frac{b}{r^3} - \frac{\theta}{\mu}\left[1+\frac{r}{k_2}+\frac{r^2}{k_2^2}\right]\frac{c}{r^3}e^{-r/k_2}\right. \\ F_2(r) &= -\left\{\frac{a}{r}-\frac{3b}{r^3}+\frac{\theta}{\mu}\left[3+3\frac{r}{k_2}+\frac{r^2}{k_2^2}\right]\frac{c}{r^3}e^{-r/k_2}\right. \end{aligned}\right\} \qquad (8.4)$$

Here $n_i = x_i/r$ is the unit radius vector. We shall re-write (8.3) in the spherical system of coords

$$v_r = \left[i + F_1(r) + F_2(r)\right]u\cos\theta, \quad v_\theta = -\left[i + F_1(r)\right]u\sin\theta, \quad v_\varphi = 0 \qquad (8.5)$$

The field of the "inherent" angular velocities Ω', in accordance with (5.11) takes the form

$$\left.\begin{aligned} &\Omega_r = 0, \quad \Omega'_\theta = 0, \quad \Omega'_\varphi = \Omega(r)\sin\theta \\ &\Omega(r) = \frac{a}{r^2} - c\left(1+\frac{r}{k_2}\right)\frac{\exp(-r/k_2)}{r^3} \end{aligned}\right\} \qquad (8.6)$$

The constants a, b, can be found from the boundary conditions in §6 ($\left[1'\right]$ + $F_1(R)=0$, $F_2(R)=0$) we find

$$a = \frac{3R}{4}\left\{1+\frac{4}{3R}c\frac{k^2}{A^2}e^{-k}\right\}, \quad b = \frac{R^3}{4}\left\{1+\frac{4}{R}\frac{(k+1)^2+1}{A^2}e^{-k}\right\} \qquad (8.7)$$

In flow 1 we use (6.6) and find

$$a = \frac{3R}{4}\left\{1+\frac{4}{3R}c\frac{k^2}{A^2}e^{-k}\right\}, \quad b = \frac{R^3}{4}\left\{1+\frac{4}{k}c\frac{(k+1)^2+1}{A^2}e^{-k}\right\} \qquad (8.8)$$

To determine the drag F experienced by the sphere, we obtain the following expression

$$F = \int_S (\sigma_{rr}\cos\theta - \sigma_{\theta r}\sin\theta)\, ds \qquad (8.9)$$

$$\sigma_{rr} = -p + 2\mu\frac{\partial v}{\partial r}; \quad \sigma_{\theta r} = \sigma^+_{\theta r} + \sigma^-_{\theta r} \qquad (8.10)$$

The latter expression contains an antisymmetrical part (γ-dependent) -. The pressure p can be found if the antisymmetric expression (5.1) be substituted into (6.2) - Hence γ^2 we obtain

$$p = -2\mu r^{-2} A(\eta_0 H_1) \qquad (8.14)$$

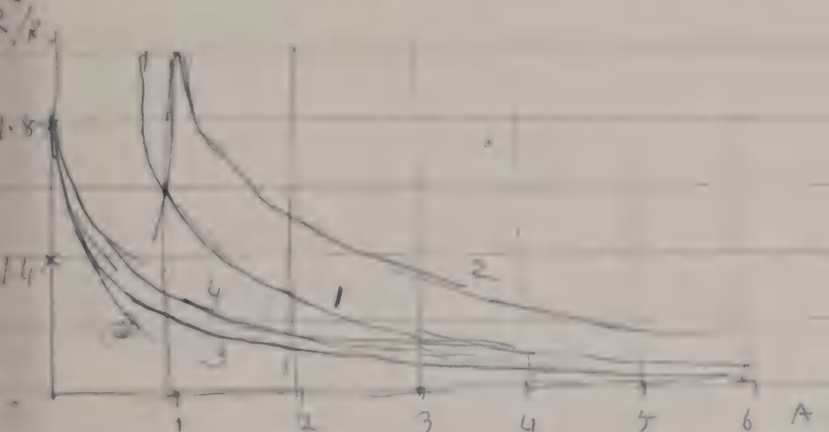
We finally find

$$F = F^0\left[1 + \frac{k^2 A^{-2}}{1 - k^2 A^{-2} + k(1+\delta_2 k)}\right], \quad F^0 = 6\pi\mu R_\mu \qquad (8.16)$$

As is seen from this expression taking the rotational friction of the fluid particles into account leads to an increase in the resistance in comparison with that given by Stokes formula - This result is formally equivalent to an increase of the hydrodynamic radius of the sphere. We shall note the limiting cases of (8.16). If

$$|\gamma|/\mu \gg 1, \text{ then } R^* = R\left[1 + A^{-1}\right], \quad (k=0)$$
$$R^* = R\left[1 + 2A^{-1}(A+B)^{-1}\right], \quad (k=0) \qquad (8.17)$$

The dependence of R^*/R on A is depicted in the fig 2. for the case $\alpha' = 0$ (curve 1) $\alpha = \infty$ (curve 2). If $|\gamma|/\mu$ is comparable to unity, the curves R^*/R will tend to a finite value determined by $|\gamma|/\mu$ as $A \to 0$. Curve 3 depicts the dependence of R^*/R on A for the case $|\gamma|/\mu = 1$ and $\alpha = 0$, and curve 4, for the case $|\gamma|/\mu = 1$ and $\alpha = 0$



(A round cell moving in a viscous fluid)

22/5/70

Coupled stresses in fluids by Thomas Brown & Ahmed S. Donnell (Refer to in the Thesis)

Abstract:- Applications of couple stress and micro-polar theories to the

to the problems of Couette & Poisseuille flows between two ||^{l} plates are discussed and the results are compared [Ref. to Eringens paper on the Micromorphic Continua and in the 1966 paper is made] [Also reference to Stokes V. J. Stokes* - Physics of Fluids 9, 1709 (1966)]

Authors consider the one-dimensional fully developed, steady flow of an incompressible fluid between two ||^{l} plates as given by Stokes and on the micromorphic theory of Eringen and results are compared.

The velocity results for the couple-stress theory given by Stokes [* Eqn (5·1a)]. on the other hand the micromorphic theory yields [Ref. to Eringen]

$$v = \frac{\gamma K}{K}\left(A\sinh ky + B\cosh ky\right) - \frac{2A}{K}\sinh ky - \frac{2B}{K}\cosh ky + \frac{2d}{2\mu+K}y$$
$$+ \frac{1}{(2\mu+K)}\left(\frac{\partial p}{\partial x}\right)y^2 + D$$
$$v = A\cosh ky + B\sinh ky - \frac{C}{2\mu+K} - \frac{1}{2\mu+K}\left(\frac{\partial p}{\partial x}\right)y \qquad (1)$$

where $K = \left(\frac{2\mu+K}{\mu+K}\cdot\frac{K}{\gamma}\right)^{1/2}$

and A, B, C, D are constants, K & γ are the new constants of viscosity of the micromorphic fluid. It has been shown by Eringen from thermodynamical considerations that γ, K and μ must be real & positive; hence must also be a real and +ve number.

The second eqn of (1) given above is not present in the couple stress theory & thus here additional viscosity coeffts are defined in the micro-fluid theory

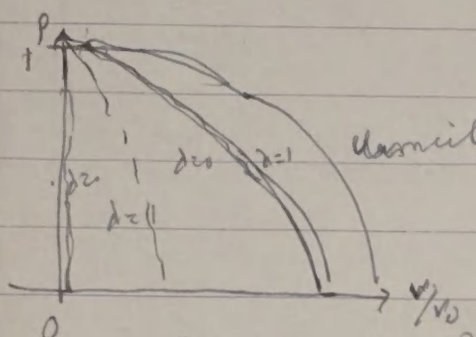
The case of Couette flow between 2 ||^{l} plates with $\partial p/\partial x = 0$ is considered for both cases & it is remarked that the boundary conds v_{xz} etc, v_{yy} etc. exist for the couple stress theory appears unreasonable. For micropolar fluid, boundary conds for the lower plate stationary and upper plate moving with const. vel. V are taken as

at $y=0$, $v=0$, $\nu=0$
at $y=h$, $v=V$, $\nu=0$ } — (3) using (3) in (1) gives expression of v/V

in terms of $\lambda \equiv Kh$ and $\xi \equiv y/h$, and also M_{yz} (Eqns (4) & (5). For $K=0$ (4) & (5) reduce to classical case of Couette flow. Vel. profiles for both theories are given in fig 1 for $\lambda = 1$ and $K/\mu = 0, \infty$

Next the case of Poiseuille flow due to a pressure gradient between two ||^{l} plates separated by $2h$ with boundary conds, $y = \pm h$, $v = 0$, $\nu = 0$

is worked out the micropolar & Stokes's theory, and velocity profiles drawn for $K/\mu = 1$ and various values of λ ranging from zero to ∞.



Similarly another typical is drawn showing the velocity profile for the case $K/\mu = \infty$

Next is considered the mass rate of flow corresponding to eqns [(5.3) & (5.4) of Stokes] for the micropolar case, and gets

$$\frac{Q}{Q_0} = 1 + \frac{3}{\lambda^2}\frac{K}{2(\mu+K)}\left(1 - \lambda \coth\lambda\right)$$

where $Q_0 = -\frac{2}{3}\rho \frac{2\lambda^3}{2\mu+K}\frac{\partial b}{\partial x}$

Again a graph of the mass-flux for various values of λ & of K/μ are drawn as compared with λ.

Stokes also solves the problem of Couette flow between two concentric cylinders with $\partial\psi/\partial\theta = 0$, and finds no difference between classical case and zero couple stress cases as in the parallel plate case. The micropolar theory, however, yields a soln which reduces to the classical case for $K/\mu = 0$

In conclusion, it is significant that the micropolar fluid theory gives two independent equations to describe the velocity and microrotation velocity of velocity fields, thus allowing the specification of both these quantities at the boundaries independently.

There is no reference in the thesis to the paper by V. J. Stokes referred to above

Physics of fluids, 9, (1966), p. 1709

Couple Stresses in fluids by Vijay Kumar Stokes (? !), I. I. T. Kanpur

(NO reference to micropolar fluids or reference to Eringen — only reference to Dahler).